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General duality for Abelian-group-valued statistical-mechanics models

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Abstract

We introduce a general class of statistical-mechanics models, taking values in an Abelian group, which includes examples of both spin and gauge models, both ordered and disordered. The model is described by a set of 'variables' and a set of 'interactions'. Each interaction is associated with a linear combination of variables; these are summarized in a matrix J. A Gibbs factor is associated with each variable (one-body term) and with each interaction. Then we introduce a duality transformation for systems in this class. The duality exchanges the Abelian group with its dual, the Gibbs factors with their Fourier transforms and the interactions with the variables. High (low) couplings in the interaction terms are mapped into low (high) couplings in the one-body terms. If the matrix J is interpreted as a vector representation of a matroid, duality exchanges the matroid with its dual. We discuss some physical examples. The main idea is to generalize the known models up to eventually include randomness into the pattern of interaction. We introduce and study a random Gaussian model, a random Potts-like model and a random variant of discrete scalar QED. Although the classical procedure as given by Kramers and Wannier does not extend in a natural way to such a wider class of systems, our weaker procedure applies to these models, too. We shortly describe the consequence of duality for each example.

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1. Introduction of the problem

In statistical mechanics a notion of duality appeared first as a relation between the high- and low-temperature expansions in the two-dimensional Ising model [1, 2]. In both expansions

the partition function can be expressed as a sum on a gas of closed polymers; in the low-temperature picture, these polymers are the boundaries of the ferromagnetic domains, so we have an expansion in the deviations from the ordered regime. In contrast, in the high-temperature picture the polymers are the terms of a Fourier expansion which expresses the deviation from a disordered regime of total decorrelation. So, in a certain sense, the duality relates *energetic* degrees of freedom to *entropic* degrees of freedom. Moreover, under the assumption of the existence of only one critical value for the temperature, it was possible to locate the phase-transition point before the exact solution of the model given by Onsager.

It was, afterwards, realized [3–8] that similar constructions hold for models with Abelian-group-valued variables.

More formally, a duality transformation can be defined directly on the partition function of the model by means of two independent duality transformations, that is:

- The variables are located on the k-cells of a D-dimensional cell-complex (e.g. the 0-cells, or *sites*, for a spin model, the 1-cells, or *bonds*, for a gauge model), more precisely the physical state is a k-cochain taking values in an Abelian group. The interactions are located on the $(k+1)$-cells (respectively the *bonds* and the *plaquettes* in the previous examples). The 'algebraic' duality exchanges variables and interactions, and the dual physical state is a $(k+1)$-chain taking values in the original group.
- The new variables have a gauge redundancy, and must obey a set of constraints. In other words, they must be a closed $(k+1)$-chain. The solution of the constraints leads to the 'integration' of the $(k+1)$-chain to a $(k+2)$-chain. After this, the cell-complex on which the model is defined is transformed into its dual counterpart. This 'geometrical' duality of cellular complexes pictorially exchanges n-cells with $(D-n)$-cells, and the physical state is now a $(D-k-2)$-cochain.

This general strategy is explained in detail, for example, in [9, 10].

These two independent transformations can also be performed in reverse order. First we have to change variables from the original set, located on the k-cells, to a new set of redundant 'gauge' variables, located on $(k+1)$-cells. We can reinterpret the original interaction terms as one-body terms, although a set of constraints located on $(k+2)$-cells appears. Now the 'geometrical' duality translates n-cells on the original lattice into $(D-n)$-cells in the dual lattice, and the physical state becomes a $(D-k-1)$-chain, while the constraints, treated as delta-function interactions, are located on $(D-k-2)$ cells. Then, the 'algebraic' duality exchanges variables and interactions, so the dual physical state is a $(D-k-2)$-cochain, and variables on each $(D-k-2)$-cell take the value of the dual of the original group.

This alternative approach [7] has the advantage of requiring a derivative of a differential form instead of a more difficult integration. Although applied mainly to models on regular lattices, this approach only requires the lattice to be of a cell complex of definite dimension D. For example, in the case of planar graphs, a duality relation can be stated for arbitrary graphs and generic patterns of interaction, for example, also in the presence of frustration [11].

This construction on the partition function has direct implications also for the expectation values of physical operators (e.g. correlation functions). For example, by considering modifications of the Hamiltonian, in [12] a 'disorder' operator $\mu(x^*)$ is identified for the 2D-Ising model, which, at zero magnetic field, has a dual role w.r.t. the spin operator $\sigma(x)$. Note that, according to the geometric procedure above, the operator μ is located on lattice plaquettes.

In this paper we wish to clarify the 'algebraic' part of the duality transformation, by putting it in a still more general context. We consider a rather general class of statistical-mechanical models in which the variables take values in an Abelian group; each interaction

term is associated with a particular linear combination of variables, but need not arise from any particular geometrical entity (such as a bond or a plaquette). By relaxing the requirement for a cell-complex structure, we can deal with a wider class of statistical models. We think that this more general notion of duality can be fruitful in the study of disordered systems. This notion of duality is motivated, in fact, by the *clauses-variables* duality that we observed in a previous investigation into the zero-temperature behaviour of a model of random satisfiability [13], which has the structure of a disordered diluted interaction.

The paper is organized as follows. In section 2 we introduce some mathematical formalism, in particular about the notion of duality and Fourier transform for generic Abelian groups. In sections 3 to 8 we discuss our duality under general aspects: in section 3 we describe the most general setting for the duality; in section 4 we specialize the model of a formulation which is manifestly suitable for statistical-mechanics models, and in section 5 we rederive the duality in this frame; in section 6 we show how to reconduct an extension of the original definition to the primitive formulation of the problem, which is reminiscent of a set of gauge fields coupled with the interaction terms; in section 7 we describe a particular extra feature appearing in the model when delta constraints appear; finally in section 8 we introduce a sufficient criterion for self-duality.

In sections 9 to 13 we show the application of these general ideas to some concrete physical examples: first, in sections 9 and 10, a sort of random Gaussian model and a variant with random constraints; further, in sections 11 and 12, a generalization of the Potts model extended to random structures of interactions and a discussion on how to recover, within our framework, the traditional results for regular lattices; finally, in section 13, we discuss a concrete example of the general setting of section 6, which is reminiscent of a scalar field theory coupled with electromagnetism, both in the case of regular lattices and of disordered systems.

In section 14 we outline some conclusions, and introduce the perspective for a further generalization of the procedure, using different mathematical structures instead of an Abelian group structure and the Fourier transform.

2. Dual for an Abelian group, Fourier transform and group homomorphisms

In this section we introduce the *Fourier transform* for locally compact Abelian groups, by means of the theory of characters (see for example [14]).

First, we introduce the concept of *dual group*. Let G be a locally compact Abelian group (we always write our Abelian groups additively). Let $\mathcal{R}_G$ be the set of irreducible representations of G. It is well known that the irreducible representations of G are all one-dimensional: thus, for each $\rho \in \mathcal{R}_G$, the representation matrix $U_\rho(g)$, is one-dimensional and coincides with the character $\chi_\rho(g)$. The set $\mathcal{R}_G$ is itself an Abelian group under ordinary composition (multiplication); we write this group additively and call it the dual group $\widehat{G}$.

We label the elements of the original group G by Latin letters $g, h, \ldots$, and elements of the dual group $\widehat{G}$ by Greek letters $\rho, \tau, \ldots$. If we adopt the notation

$$\chi_\rho(g) = U_\rho(g) = \mathrm{e}^{\mathrm{i}\vartheta_G(\rho,g)} \tag{1}$$

the following bilinear relations hold:

$$\vartheta_G(\rho+\tau, g) = \vartheta_G(\rho, g) + \vartheta_G(\tau, g) \tag{2a}$$

$$\vartheta_G(\rho, g+h) = \vartheta_G(\rho, g) + \vartheta_G(\rho, h). \tag{2b}$$

The dual group of $\widehat{G}$, denoted by $\widehat{\widehat{G}}$, is naturally isomorphic to G.

In a consistent choice of notation, the characters of the dual group are derivable from the antisymmetric relation

$$\vartheta_G(\rho, g) = -\vartheta_{\widehat{G}}(g, \rho) \tag{3}$$

which is equivalent to

$$\chi_g(\rho) = \overline{\chi_\rho(g)}. \tag{4}$$

Given a complex-valued function f on G, its Fourier transform $\widehat{f}$ is a function on the dual group $\widehat{G}$, defined by

$$f : G \to \mathbb{C} \qquad f(g) = \int_\rho^* \mathrm{e}^{\mathrm{i}\vartheta_{\widehat{G}}(g,\rho)}\, \widehat{f}(\rho) \tag{5a}$$

$$\widehat{f} : \widehat{G} \to \mathbb{C} \qquad \widehat{f}(\rho) = \int_g \mathrm{e}^{\mathrm{i}\vartheta_G(\rho,g)} f(g). \tag{5b}$$

Notation $\int_g$ and $\int_\rho^*$ stand for a properly normalized Haar measure on G and $\widehat{G}$. This allows us to introduce a notion of invariant δ-function compatible with the measure introduced above:

$$\int_\rho^* \overline{\chi_\rho(g')}\chi_\rho(g) = \delta_G(g' - g) \qquad \int_g \overline{\chi_{\rho'}(g)}\chi_\rho(g) = \delta_{\widehat{G}}(\rho' - \rho)$$

which have the natural property

$$\int_g f(g)\delta_G(g' - g) = f(g') \qquad \int_\rho^* \widehat{f}(\rho)\delta_{\widehat{G}}(\rho' - \rho) = \widehat{f}(\rho').$$

Subscripts G and $\widehat{G}$ will be omitted when there is no confusion.

Also a convolution operation can be introduced in this general context. Indeed, define the convolution product as

$$(f_1 * f_2)(g) = \int_{g'} f_1(g') f_2(g - g') \tag{6}$$

$$(\widehat{f}_1 * \widehat{f}_2)(\rho) = \int_{\rho'}^* \widehat{f}_1(\rho')\widehat{f}_2(\rho - \rho') \tag{7}$$

then the property

$$(\widehat{f}_1 * \widehat{f}_2)(\rho) = \widehat{f_1 f_2}(\rho) \tag{8}$$

follows[3].

The most general Abelian group with the properties of being locally compact and compactly generated (which are natural properties when describing physical variables) is

[3] Consider the following algebra, in which linearity property (2) and the δ-function definition are used:

$$\begin{aligned}(\widehat{f}_1 * \widehat{f}_2)(\rho) &= \int_{\rho'}^* \widehat{f}_1(\rho')\widehat{f}_2(\rho - \rho') = \int_{\rho'}^* \int_{g_1}\int_{g_2} f_1(g_1) f_2(g_2)\, \mathrm{e}^{\mathrm{i}(\vartheta(\rho', g_1) + \vartheta(\rho - \rho', g_2))} \\ &= \int_{\rho'}^* \int_{g_1}\int_{g_2} f_1(g_1) f_2(g_2)\, \mathrm{e}^{\mathrm{i}(\vartheta(\rho', g_1 - g_2) + \vartheta(\rho, g_2))} = \int_{g_1}\int_{g_2} f_1(g_1) f_2(g_2)\, \mathrm{e}^{\mathrm{i}\vartheta(\rho, g_2)}\delta(g_1 - g_2) \\ &= \int_g f_1(g) f_2(g)\, \mathrm{e}^{\mathrm{i}\vartheta(\rho, g)} = \widehat{f_1 f_2}(\rho).\end{aligned}$$

Table 1. Convention for the normalization of the Haar measure adopted in this paper for various pairs $(G, \widehat{G})$ of Abelian groups. It is understood that for the direct product of groups the Haar measure is simply the product measure of the original ones.

	$(\mathbb{Z}_q; \mathbb{Z}_q)$	$(U(1); \mathbb{Z})$	$(\mathbb{R}; \mathbb{R})$
$\int_g$:	$\frac{1}{q}\sum_{x=0}^{q-1}$	$\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}$	$\int_{-\infty}^{\infty} \mathrm{d}x$
$\int_\rho^*$:	$\sum_{\xi=0}^{q-1}$	$\sum_{n=-\infty}^{+\infty}$	$\int_{-\infty}^{\infty} \mathrm{d}\xi$
$\vartheta(g, \rho)$:	$2\pi x\xi/q$	θn	$2\pi x\xi$
$\delta_G(g)$:	$q\delta_{\mathrm{Kr}}(x)$	$2\pi\delta_{\mathrm{Dirac}}(\theta)$	$\delta_{\mathrm{Dirac}}(x)$
$\delta_{\widehat{G}}(\rho)$:	$\delta_{\mathrm{Kr}}(\xi)$	$\delta_{\mathrm{Kr}}(n)$	$\delta_{\mathrm{Dirac}}(\xi)$

the product of $\mathbb{R}, U(1), \mathbb{Z}$ and $\mathbb{Z}_q$ groups[4], so the suggested choice of normalizations of table 1 covers all the possible situations.

A homomorphism φ from an Abelian group G_1 to another Abelian group G_2 is a map which preserves the group operation $\varphi(x +_{G_1} x') = \varphi(x) +_{G_2} \varphi(x')$, and maps the identity to the identity, $\varphi(0) = 0$. We denote by $\mathrm{Hom}(G_1, G_2)$ the set of all homomorphisms from G_1 to G_2. Note that a homomorphism must preserve additive inverses, i.e. $\varphi(-g) = -\varphi(g)$. This follows from $\varphi(0) = \varphi(g) + \varphi(-g) = 0$.

As we have seen in the previous paragraphs, from the theory of characters we can introduce the dual groups $\widehat{G}_1$ and $\widehat{G}_2$. It is indeed possible to introduce the notion of dual homomorphism[5], $\varphi^T \in \mathrm{Hom}(\widehat{G}_2, \widehat{G}_1)$, which completes the diagram

$$\begin{array}{ccc} G_1 & \prec\!\!-\!\!-\!\!\stackrel{\mathcal{D}}{-}\!\!-\!\!\succ & \widehat{G}_1 \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\uparrow{\scriptstyle\varphi^T} \\ G_2 & \prec\!\!-\!\!-\!\!\stackrel{\mathcal{D}}{-}\!\!-\!\!\succ & \widehat{G}_2 \end{array}$$

The dual homomorphism φ^T acts according to the formula

$$\chi_\rho(\varphi(g)) = \chi_g(\varphi^T(\rho)) = \overline{\chi_{\varphi^T(\rho)}(g)} \qquad g \in G_1 \qquad \rho \in \widehat{G}_2 \tag{9}$$

from which we derive

$$\vartheta_{G_2}(\varphi(x), y) = \vartheta_{G_1}(\varphi^T(y), x). \tag{10}$$

It is easy to verify that φ^T is a homomorphism from $\widehat{G}_2$ to $\widehat{G}_1$, as a consequence of the fact that $\vartheta(\rho, g)$ is a bilinear form (2)

$$\begin{aligned} \chi_{\varphi^T(\rho_1+\rho_2)}(g) &= \overline{\chi_{\rho_1+\rho_2}(\varphi(g))} = \overline{\chi_{\rho_1}(\varphi(g))} \cdot \overline{\chi_{\rho_2}(\varphi(g))} \\ &= \chi_{\varphi^T(\rho_1)}(g) \cdot \chi_{\varphi^T(\rho_2)}(g) = \chi_{\varphi^T(\rho_1)+\varphi^T(\rho_2)}(g). \end{aligned} \tag{11}$$

[4] This fact is derived essentially from the three general results [15]:

- Every locally compact, compactly generated Abelian group G is topologically isomorphic with $\mathbb{R}^a \times \mathbb{Z}^b \times K$ for some non-negative integers a and b and some compact Abelian group K.
- Let K be a compact Abelian group and let U be a neighbourhood of the identity in K. There is a closed subgroup H of G such that $H \subset U$ and K/H is topologically isomorphic with $U(1)^c \times F$, where c is a non-negative integer and F is a finite Abelian group.
- Let F be a finite Abelian group. Then it is isomorphic to the group direct product of cyclic groups of prime power order (*Kronecker decomposition theorem*).

[5] For example cf Kirillov [14], Part III, statement 631.

3. The general formulation of the duality

In this section we wish to introduce a notion of duality for a wide class of systems in classical statistical mechanics.

In the most general form, given two Abelian groups G_1 and G_2, we will consider the system defined by

- a function $F_1 : G_1 \to \mathbb{C}$;
- a function $\widehat{F}_2 : \widehat{G}_2 \to \mathbb{C}$;
- a homomorphism $\varphi \in \mathrm{Hom}(G_1, \widehat{G}_2)$;

and the Gibbs weight of a state $x \in G_1$ is given by

$$\exp\left[-\mathcal{H}_{F_1,\widehat{F}_2;\varphi}(x)\right] = F_1(x)\widehat{F}_2(\varphi(x)). \tag{12}$$

The partition function is defined as

$$Z_{F_1,\widehat{F}_2;\varphi} = \int_x F_1(x)\widehat{F}_2(\varphi(x)). \tag{13}$$

Applying Fourier transform to the function $\widehat{F}_2$ we obtain an algebraic relation

$$\begin{aligned}\int_x F_1(x)\widehat{F}_2(\varphi(x)) &= \int_x\int_y F_1(x)F_2(y)\,\mathrm{e}^{\mathrm{i}\vartheta_{G_2}(\varphi(x),y)} \\ &= \int_y\int_x F_2(y)F_1(x)\,\mathrm{e}^{\mathrm{i}\vartheta_{G_1}(\varphi^T(y),x)} = \int_y F_2(y)\widehat{F}_1(\varphi^T(y))\end{aligned}$$

from which we deduce that

$$Z_{F_1,\widehat{F}_2;\varphi} = Z_{F_2,\widehat{F}_1;\varphi^T}. \tag{14}$$

Note that in the dual model the roles of the groups G_1 and G_2 are interchanged, and the homomorphism $\varphi \in \mathrm{Hom}(G_1, \widehat{G}_2)$ is replaced by $\varphi^T \in \mathrm{Hom}(G_2, \widehat{G}_1)$.

4. Definition of a model with many degrees of freedom

In this section we specialize the procedure described above to a more physical frame, in which *extensive* parameters appear both in the choice of the groups G_1 and $\widehat{G}_2$ and of the functions F_1 and $\widehat{F}_2$.

We introduce two such parameters: the first one, r, is called *the number of variables*, while the second one, $(n-r)$, is *the number of interactions* (they were called *clauses* in [13]).

Given a group G, we adopt $G_1 = G^{\otimes r}$ and $\widehat{G}_2 = G^{\otimes(n-r)}$. A state in G_1 is specified by a vector $x = (x_1, \ldots, x_r)$, and the homomorphism φ is specified by a $r \times (n-r)$ matrix J, with elements $J_{ik} \in \mathrm{Hom}(G, G)$ acting on the left, such that $(xJ)_k$ means $\sum_{i=1}^{r} J_{ik}(x_i)$. By convention we let the indices k run from $r+1$ to n.

The functions F_1 and $\widehat{F}_2$ will be chosen to be factorized on the entries of the vectors:

$$F_1[x] = \prod_{i=1}^{r} f_i(x_i) \qquad \widehat{F}_2[y] = \prod_{k=r+1}^{n} f_k(y_k). \tag{15}$$

The weight functions $f_1, \ldots, f_r$ are thus 'one-body terms' (generalized magnetic fields), while $f_{r+1}, \ldots, f_n$ are 'interaction terms'.

Note that we have chosen the groups G_1 and $\widehat{G}_2$ to be direct products of *the same* group G. To avoid confusion in the application of the conventions of table 1, we specify that

$$\int_{[x]\in G_1} \equiv \int_{x_1} \cdots \int_{x_r} \qquad \vartheta_{G_1}(x', x) = \sum_i \vartheta_G(x'_i, x_i)$$
$$\int^*_{[y]\in G_2} \equiv \int^*_{y_{r+1}} \cdots \int^*_{y_n} \qquad \vartheta_{G_2}(y', y) = -\sum_k \vartheta_G(y_k, y'_k).$$

In this frame, we can develop a physical intuition on the structure of our model. We could consider the r entries of the vector x as the variables of a statistical-mechanics system. The first r functions $f_i(x_i)$ collect the one-body contributions to the Gibbs factor, like one-body measure terms, or magnetic fields, or chemical potentials. The last $(n-r)$ functions $f_k((xJ)_k)$ act like $(n-r)$ terms of interaction, encoding information like the strength of the couplings, or the nature of the clauses, etc.

The matrix J describes how the variables interact. As an example take the entries of J in a commutative ring with unit, $\mathcal{R}$, and G an $\mathcal{R}$-module. A particular case occurs when G is an arbitrary Abelian group and $\mathcal{R} = \mathbb{Z}$. In fact, every additive group is an $\mathcal{R}$-module with $\mathbb{Z}$, where $n \cdots g = g + \cdots + g$ (n times). As an example consider a scalar field theory in which the r variables ϕ_i take a real value for each site i of the lattice, and the interaction terms depend on $(\phi_i - \phi_j)^2$, where (i, j) is a bond of the lattice. Another example is a gauge Potts model, in which the r variables s_i take a value in $\mathbb{Z}_q$ for each bond i of the lattice, and the interaction terms depend on the linear combinations $\sum_{i\in\partial p} s_i$, where p is a plaquette of the lattice. In fact in both these models all the coefficients of J are in $\{0, \pm 1\}$. In the first case, the entry J_{ij} is nonzero when the site labelled by i is an extremum of the edge labelled by j, while in the second case this happens when the edge labelled by i is on the border of the plaquette labelled by j, and the sign is determined by the choice of orientation for edges and plaquettes.

As a second example take the entries of J in a field $\mathbb{F}$, and G a finite-dimensional vector space over $\mathbb{F}$ (say, $\mathbb{F}^l$). The field $\mathbb{F}$ could be, for example, the real field $\mathbb{R}$, the complex field $\mathbb{C}$, or a finite field $GF(q^k)$.

Moreover, we stress the fact that the generality of our formulation does not require any underlying geometrical structure for the system, and also applies to situations typical of disordered mean-field models, in which such a structure does not exist.

Denote with 1 the identity homomorphism $\varphi_I \in \mathrm{Hom}(G, G)$, such that $\varphi_I(x) = x$ for any x, and with 0 the null homomorphism $\varphi_0 \in \mathrm{Hom}(G, G)$, such that $\varphi_0(x) = 0$ for any x. We can introduce an $r \times n$ matrix B defined as

$$B_{ij} = \begin{cases} 1 & i = j \quad 1 \leqslant j \leqslant r \\ 0 & i \neq j \quad 1 \leqslant j \leqslant r \\ J_{ij} & r+1 \leqslant j \leqslant n \end{cases} \tag{16}$$

i.e. in short notation

$$B = [I_r | J]. \tag{17}$$

This allows us to use a unique label for the two functions $F_1[x]$ and $\widehat{F}_2[xJ]$, defining $F[xB] = F_1[x]\widehat{F}_2[xJ] = \prod_{j=1}^n f_j((xB)_j)$.

The invariance of the partition function under change of variables in the integration corresponds in terms of the matrix B to the application of a special orthogonal transformation from the left (*change of basis*)

$$Z[B] = \int_{G^r} F[xB] \qquad Z[B] = Z[OB] \qquad \forall O \in SO(r). \tag{18}$$

The result is that, beyond the particular choice of the matrix B, a more fundamental notion of 'pattern of interaction' can be established, which is independent of the choice of basis. This is the analogous of what happens in linear algebra, where the notion of linear independence is intrinsic with respect to the choice of basis. The introduction of this concept in mathematics dates up to the paper of Whitney [16], and is the foundation of a theory called *matroid theory* (cf [17] for a textbook). In particular, *matroid* is the name given to the intrinsic mathematical structure corresponding to our pattern of interaction, and a choice for the matrix B is called a *vectorial representation* for the matroid. In the case in which the first $r \times r$ block of the matrix is the identity, we say that B is a *standard vectorial representation* for the matroid. Note that the definition of our model puts B automatically in standard vectorial form.

5. Duality for product groups

Let us now repeat the procedure of the previous section specialized to our case, and write the corresponding duality transformation. The Boltzmann weight is

$$\exp(-\mathcal{H}_{G;f;B}(x)) = \prod_{j=1}^{n} f_j((xB)_j). \tag{19}$$

The partition function is

$$Z_{G;f;B} = \int_{[x]} \exp(-\mathcal{H}_{G;f;B}(x)) \tag{20}$$

which, applying Fourier transform, can be restated as

$$\begin{aligned} Z_{G;f;B} &= \int_{[x]} \prod_{i=1}^{r} f_i(x_i) \prod_{k=r+1}^{n} f_k((xJ)_k) = \int_{[x]} \int_{[y]}^{*} \mathrm{e}^{\mathrm{i}\vartheta_{G_2}(xJ,y)} \prod_{i=1}^{r} f_i(x_i) \prod_{k=r+1}^{n} \widehat{f}_k(y_k) \\ &= \int_{[y]}^{*} \int_{[x]} \mathrm{e}^{\mathrm{i}\vartheta_{G_1}(-yJ^T,x)} \prod_{k=r+1}^{n} \widehat{f}_k(y_k) \prod_{i=1}^{r} f_i(x_i) \\ &= \int_{[y]}^{*} \prod_{k=r+1}^{n} \widehat{f}_k(y_k) \prod_{i=1}^{r} \widehat{f}_i((-yJ^T)_i) \end{aligned}$$

from which we deduce that

$$Z_{G;f;B} = Z_{\widehat{G};\widehat{f};\widehat{B}} \tag{21}$$

where the dual $\widehat{B}$ of the matrix B is understood exactly as in matroid theory (see [17], chapter 2).

$$B = [I_r|J] \qquad \widehat{B} = [-J^T|I_{n-r}]. \tag{22}$$

In the most general case of $J_{ij} \in \mathrm{Hom}(G, G)$, the quantities $(J^T)_{ji}$ are the dual homomorphisms $\widehat{J}_{ij}$ defined according to (9).

The duality interchanges the role of variables and interactions. Furthermore, if the functions f_j are almost constant (low coupling, or high temperature), the functions $\widehat{f}_j$ are very peaked (high coupling, or low temperature), so the duality relates a regime of high (resp. low) temperature in the one-body terms with a regime of low (resp. high) temperature in the interaction terms, and vice versa.

The analogous Kadanoff–Ceva procedure [12], referred to in the introduction, which relates the expectation values of physical operator pairs as a consequence of duality, can also be repeated in this general case, although a little care is required.

Consider a local operator

$$A(x) = A_{i_1}\big(x_{i_1}\big)\cdots A_{i_l}\big(x_{i_l}\big)\cdot A_{k_1}\big((xJ)_{k_1}\big)\cdots A_{k_m}\big((xJ)_{k_m}\big) \tag{23}$$

whose expectation value can be expressed as the ratio of two partition functions, the numerator corresponding to a properly modified system

$$\langle A\rangle_{G;f;B} = \frac{Z_{G;f^{(A)};B}}{Z_{G;f;B}} \qquad f_j^{(A)}(z) := \begin{cases} f_j(z)A_j(z) & j \in \{i_1,\ldots,i_l,k_1,\ldots,k_m\} \\ f_j(z) & \text{otherwise.} \end{cases} \tag{24}$$

This expectation value coincides with the expectation value of a related operator in the dual system. The proper choice is *not* the naive one $\widehat{A}$, but a certain A^* defined as

$$A_j^*(\xi) := \frac{(\widehat{f}_j * \widehat{A}_j)(\xi)}{\widehat{f}_j(\xi)} \qquad j \in \{i_1,\ldots,i_l,k_1,\ldots,k_m\} \tag{25}$$

for which we have the relation

$$Z_{G;f^{(A)};B} = Z_{\widehat{G};\widehat{f}^{(A^*)};\widehat{B}} \quad \Longrightarrow \quad \langle A\rangle_{G;f;B} = \langle A^*\rangle_{\widehat{G};\widehat{f};\widehat{B}}. \tag{26}$$

Note that local operators acting on variables are mapped into local operators acting on interactions, and vice versa.

6. The coupling with gauge fields

In this section we slightly generalize the formulation of the problem introduced in sections 3 and 4, and then we show that, with a change of variables, it can be recast in the original framework.

This analysis has one main reason: in section 13 we will describe the physical examples both of a scalar field theory, and of a discretized free electrodynamic theory, and show that our duality holds for both systems. Then we show that the duality also holds for the system in which the electrodynamics is coupled with the scalar field in the traditional gauge-covariant way.

We want to show that this is a particular case of a general recipe, introduced in this section, and valid for the cases in which we introduce a set of gauge fields coupled with the terms of interaction. The change of variables corresponds to converting the basis of scalar fields and gauge fields into the basis of fields and covariant derivatives.

Given $2k$ Abelian groups $G_1,\ldots,G_k,G'_1,\ldots,G'_k$, we will consider the system defined by

- k functions f_i from the group G_i to the field $\mathbb{C}$;
- k functions $\widehat{g}_i$ from the group $\widehat{G}'_i$ to the field $\mathbb{C}$;
- $2k-1$ homomorphisms φ_i between the groups G_i and $\widehat{G}'_i$:

$$G_k \xrightarrow{\varphi_{-k+1}} \cdots \xrightarrow{\varphi_{-1}} G_1 \xrightarrow{\varphi_0} \widehat{G}'_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{k-1}} \widehat{G}'_k \tag{27}$$

and the Gibbs weight of a state $x = (x_1,\ldots,x_k) \in \bigotimes_i G_i$ is

$$\exp\big[-\mathcal{H}_{f,\widehat{g};\varphi_i}(x)\big] = f_k(x_k)\prod_{i=1}^{k-1} f_i(x_i - \varphi_{-i}(x_{i+1}))\prod_{j=1}^{k}\widehat{g}_j(\varphi_{j-1}\cdots\varphi_0(x_1)). \tag{28}$$

Consider the change of variables $y_i = x_i - \varphi_{-i}(x_{i+1})$, with Jacobian 1. If we introduce the $k\times k$ matrix of homomorphisms

$$\psi_{ij} = \varphi_{-(i-1)}\varphi_{-(i-2)}\cdots\varphi_{j-1} \tag{29}$$

where the homomorphism ψ_{ij} goes from G_i to $\widehat{G}'_j$, we could restate the Gibbs weight above as

$$\exp\left[-\mathcal{H}_{f,\widehat{g};\varphi_i}(y)\right] = \prod_{i=1}^{k} f_i(y_i) \prod_{j=1}^{k} \widehat{g}_j \left(\sum_{l=1}^{k} \psi_{lj}(y_l) \right). \tag{30}$$

As a matrix of homomorphisms from the set of groups $\{G_i\}$ to the set of groups $\{\widehat{G}'_j\}$ is a homomorphism between the product groups $\bigotimes_i G_i$ and $\bigotimes_j \widehat{G}'_j$, we are in a position to apply the general procedure outlined in section 3.

The dual model interchanges the groups G_i with the groups G'_i, and the functions f_i with the functions g_i. As the transposed homomorphisms are given by

$$(\psi^T)_{ij} = (\psi_{ji})^T = \varphi^T_{i-1} \cdots \varphi^T_{-(j-1)} \tag{31}$$

it corresponds to interchanging the homomorphism φ_i with the transposal homomorphism φ^T_{-i}. In this specialization of the general procedure, the relation (14) reads

$$Z_{f,\widehat{g};\{\varphi_i\}} = Z_{g,\widehat{f};\{\varphi^T_{-i}\}}. \tag{32}$$

In the case that we deal with product groups, as described in section 4, the matrix B will not still be in the standard form representation of matroid theory when expressed in the x basis. Nevertheless, it will be in block form with k rows and $2k$ columns. If we call J_i the linear applications corresponding to the homomorphism φ_i, the block B_{ij} is given by

$$B_{ij} = \begin{cases} I & i = j \leqslant k \\ -J_{-(k-i)} & i = j+1 \leqslant k \\ J_0 J_1 \cdots J_{j-k-1} & i = k \quad j > k \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

The matrix B can be easily diagonalized. This results in the change of basis from x variables to y variables in the sense described above. The resulting matrix B' will have blocks B'_{ij} given by

$$B'_{ij} = \begin{cases} I & i = j \leqslant k \\ 0 & i \neq j \quad j \leqslant k \\ J_{-i+1} J_{-i+2} \cdots J_{j-k-1} & j > k. \end{cases} \tag{34}$$

Now that B' is in the standard form representation, we can apply duality, and then, inverting the diagonalization procedure, restate the dual problem in the original block form. The transformation interchanges the matrices J_i with J^T_{-i}, with a minus sign on J_0, which goes into $-J_0^T$.

7. The solution of delta constraints

A general feature of our duality in the factorized case is that the resolution of constraints taking the form of delta functions has a dual counterpart in the removal of trivial constant factors, and vice versa. This feature can be seen in a transparent way in a simple unstructured problem, discussed in section 10, but it is more commonly used when the duality is combined with other geometric properties of the systems, such as, for example, in the case of models defined over regular finite-dimensional cell-complexes. Such a situation will be described in section 12.

Consider a system with r variables and $n - r$ interactions, in the standard form described above, in the case in which the first r functions $f_i(x)$ are constant. For simplicity, say that

the last $(n-r)$ functions $f_k(y)$ are all equal to a certain $f(y)$. The model is described by the Gibbs weight

$$\exp[-\mathcal{H}(x)] = \prod_{j=r+1}^{n} f((xJ)_j). \tag{35}$$

Following the duality procedure, the dual Gibbs weight is given by

$$\exp[-\widehat{\mathcal{H}}(\xi)] = \prod_{i=1}^{r} \delta((-\xi J^T)_i) \prod_{j=r+1}^{n} \widehat{f}(\xi_j). \tag{36}$$

In the following, assume for simplicity that the incidence matrix J has maximal rank[6]. In this case it can be written in standard form $J = (I, J')$ by means of a change of variables, and a relabelling of the last $(n-r)$ indices. So we can rewrite the Gibbs weights above as

$$\exp[-\mathcal{H}(x)] = \prod_{j=r+1}^{2r} f(x_j) \prod_{j=2r+1}^{n} f((xJ')_j) \tag{37}$$

$$\exp[-\widehat{\mathcal{H}}(\xi)] = \prod_{i=1}^{r} \delta\big(-\xi_i^{\mathrm{dep}} - (\xi^{\mathrm{ind}} J'^T)_i\big) \prod_{j=r+1}^{2r} \widehat{f}\big(\xi_j^{\mathrm{dep}}\big) \prod_{j=2r+1}^{n} \widehat{f}\big(\xi_j^{\mathrm{ind}}\big) \tag{38}$$

where in the last expression we already stressed that the variables ξ^{ind} form an independent set, while the variables ξ^{dep} can be expressed in terms of them when solving the delta constraints. In this case we obtain

$$\exp[-\widehat{\mathcal{H}}(\xi)] = \prod_{j=r+1}^{2r} \widehat{f}((-\xi^{\mathrm{ind}} J'^T)_j) \prod_{j=2r+1}^{n} \widehat{f}\big(\xi_j^{\mathrm{ind}}\big). \tag{39}$$

This is exactly the Gibbs weight obtained by applying duality directly to the weight in (37), considered as a system with r variables, labelled with the indices from $r+1$ to $2r$, and $n-2r$ interactions, labelled with the indices from $2r+1$ to n.

In conclusion, we draw a scheme which summarizes the procedure described above:

$$\begin{array}{ccc}
\begin{bmatrix} I, & I, & J \\ 1 & & f \end{bmatrix} & \xrightarrow{\text{remove constant factors}} & \begin{bmatrix} I, & J \\ & f \end{bmatrix} \\
\uparrow\downarrow \text{ Duality} & & \uparrow\downarrow \text{ Duality} \\
\begin{bmatrix} -I & , & I & 0 \\ -J^T & , & 0 & I \\ \delta & & & \widehat{f} \end{bmatrix} & \xrightarrow{\text{solve the } \delta} & \begin{bmatrix} -J^T, & I \\ & \widehat{f} \end{bmatrix}.
\end{array}$$

8. A sufficient condition for self-duality

Duality is particularly useful whenever it is possible to recover that a system, under suitable conditions, is self-dual.

[6] Matrices of non-maximal rank $r' < r$ would only imply trivial overall factors of the kind $|\mathbb{F}|^{(r-r')}$, or the necessity of fixing 'by hand' $(r-r')$ variables, as for example in the planar Potts model, where a factor q originates from this fact, or in the XY-model–Coulomb gas relation, in which we should avoid summing over the variable associated with one special site, in order to fix a 'ground' for the potential.

In the following, let us denote by X the instance of a problem (say, the lattice, the set of couplings, ...), and by σ a configuration of the system (say, the values of all the spins in the system).

Duality means that there is an involution $\{X\} \xrightarrow{\iota} \{X\}$ such that

$$Z[\iota X] = Z[X]e^{C} \tag{40}$$

with C an extensive quantity depending in an analytic way on the thermodynamic parameters of the system.

Of course a sufficient condition for an instance X^* to be self-dual is that it is a fixed point of the involution

$$\iota X^* = X^*. \tag{41}$$

If there is a group of invariances $\mathcal{U}$ for the partition function, i.e. $Z[uX] = Z[X]$ for all $u \in \mathcal{U}$, then a weaker condition for self-duality can be stated. It is sufficient that some $u \in \mathcal{U}$ exists such that

$$u\iota X^* = X^*. \tag{42}$$

In the context of disordered systems the variable X describing the instance is itself a random variable, distributed over some space $\{X\}$ with some normalized measure $\mu(X)$ given *a priori*. The action of duality on the space of instances $\{X\}$ naturally induces an action on the space of normalized measures over $\{X\}$

$$\mu_\iota(X) = \left|\frac{\partial \iota X}{\partial X}\right|^{-1} \mu(X). \tag{43}$$

The condition for self-duality now applies to measures. The trivial condition for a measure $\mu^*(X)$ to define a self-dual problem is

$$\mu^*_\iota(X) = \mu^*(X). \tag{44}$$

If there is a group $\mathcal{U}$ of invariances for the problem, a weaker condition can be stated on central projections:

$$\int_{\mathcal{U}} \mu^*_\iota(uX) = \int_{\mathcal{U}} \mu^*(uX) \tag{45}$$

with $\int_{\mathcal{U}}$ a normalized Haar measure on the group.

In the case of a disordered system in our general form, we have a measure on the space of n-tuples of functions $\{f_i\}$ and of $r \times (n-r)$ incidence matrices J. For $r = n/2$, all the first r functions f_i equal to a certain $f(x)$ and all the last r functions f_k equal to Fourier transform $\widehat{f}(x)$, the dual set of functions coincides with the original one, and self-duality needs to be checked only on the measure $\mu(J)$ for the incidence matrix J.

Call $A \subseteq \mathrm{Aut}(G)$ the set of automorphisms for the functions $f(x)$ and $\widehat{f}(x)$, then a generic element u of the group of invariance $\mathcal{U}$ for the model is of the form $u = (\pi; \{a_j\})$ with $\pi \in \mathfrak{S}_{\{1,\ldots,r\}} \times \mathfrak{S}_{\{r+1,\ldots,n\}}$,[7] and $a_j \in A$, with $j = 1, \ldots, n$. The action of u on a matrix J is

$$(uJ)_{ik} = a_i J_{\pi(i)\pi(k)} a_k^{-1} \qquad i \in 1, \ldots, r \quad k \in r+1, \ldots, n \tag{46}$$

and group composition is given by

$$u'u = (\pi'\pi; \{a'_j a_{\pi'(j)}\}). \tag{47}$$

The condition for self-duality (45) above is verified under many sufficiency hypotheses. For example, if both $f(x)$ and $\widehat{f}(x)$ are even, $(-1) \in A$ and central projection is sign-invariant, $\left(\int_{\mathcal{U}} f(uJ) = \int_{\mathcal{U}} f(u(-J))\right)$. In this case a sufficient condition is $\mu(J) = \mu(J^T)$, which is verified in many natural ensembles, e.g., when the entries J_{ij} are i.i.d., or when J describes the adjacency over a random bipartite graph, with balanced distributions of coordinations over the vertices of the two kinds.

[7] $\mathfrak{S}_\Lambda$ denotes the permutation group on the set Λ.

9. An easy example: the Gaussian model

As a first easy application of formula (21) we will consider a simple model exactly solvable via Gaussian integration. Consider the system described by equation (19), specialized to the case in which $G = \mathbb{R}$ and the functions $\{f_j\}$ are

$$f_i(x) = \mathrm{e}^{-x^2/2\sigma_i^2} \qquad \text{for} \quad i = 1, \ldots, r \tag{48a}$$

$$f_{r+k}(x) = \mathrm{e}^{-x^2/2\rho_k^2} \qquad \text{for} \quad k = 1, \ldots, n-r. \tag{48b}$$

We will introduce an *ad hoc* notation to label the partition function of this system

$$Z_{\mathrm{Gauss}}(J; \{\sigma_i\}, \{\rho_k\}) \equiv Z_{\mathbb{R},B,\{f_j\}} \tag{49}$$

where expressions (48) above for the $\{f_j\}$ and the vectorial representation $B = [I_r|J]$ are understood.

We could rewrite equation (19) in the more familiar form

$$\exp(-\mathcal{H}_{G;f;B}(x)) = \prod_{j=1}^{n} f_j((xB)_j) = \mathrm{e}^{-\frac{1}{2}x^T(S^{-2}+J^T R^{-2} J)x} \tag{50}$$

where $S_{ii'} = \sigma_i \delta_{ii'}$ and $R_{kk'} = \rho_k \delta_{kk'}$. Using the convention of table 1 for the measure $\int_x$ in the case $X = \mathbb{R}$ (that is, writing down the proper number of (2π) factors), and applying the formula for Gaussian integration, we have for the partition function

$$Z_{\mathrm{Gauss}}(J; \{\sigma_i\}, \{\rho_k\}) = \int \mathrm{d}x\, \mathrm{e}^{-\frac{1}{2}x^T(S^{-2}+J^T R^{-2} J)x} = \frac{(2\pi)^{\frac{r}{2}} \det S}{\sqrt{\det(I_r + M^T M)}} \tag{51}$$

where we defined

$$M = R^{-1} J S. \tag{52}$$

From equation (21) we have that the dual partition function is

$$Z_{\mathrm{Gauss}}\left(-J^T; \left\{\rho_k^{-1}\right\}, \left\{\sigma_i^{-1}\right\}\right) = \frac{(2\pi)^{\frac{n-r}{2}} \det R^{-1}}{(2\pi)^{\frac{r}{2}} \det S} Z_{\mathrm{Gauss}}(J; \{\sigma_i\}, \{\rho_k\}) \tag{53}$$

while, applying the formula for Gaussian integration on the dual formulation,

$$Z_{\mathrm{Gauss}}(-J^T; \{\rho_k^{-1}\}, \{\sigma_i^{-1}\}) = \int \mathrm{d}\eta\, \mathrm{e}^{-\frac{1}{2}\eta^T(R^2+J^T S^2 J)\eta} = \frac{(2\pi)^{\frac{n-r}{2}} \det R^{-1}}{\sqrt{\det(I_{n-r} + MM^T)}}. \tag{54}$$

So the consequences of the duality can be restated in the algebraic equation

$$\det(I_r + M^T M) = \det(I_{n-r} + MM^T) \tag{55}$$

which is indeed true, although not totally trivial[8].

[8] Consider the formal power series in z:

$$\ln(\det(I - zM^T M)) = \mathrm{tr}(\ln(I - zM^T M)) = -\sum_{n=1}^{+\infty} \frac{z^n}{n} \mathrm{tr}(M^T M)^n$$
$$= -\sum_{n=1}^{+\infty} \frac{z^n}{n} \mathrm{tr}(MM^T)^n = \mathrm{tr}(\ln(I - zMM^T)) = \ln(\det(I - zMM^T)).$$

10. A randomly constrained Gaussian model

In this section we introduce a model in which some Gaussian distributed variables in $\mathbb{R}^D$ are subjected to a set of random constraints.

This model is a first example in which the matrix of the homomorphisms J really contains non-trivial group homomorphisms. Furthermore, it exhibits in a clear way the general feature described in section 7: the resolution of delta constraints has a dual counterpart in the removal of trivial constant factors.

The model is defined as follows. Let $x_i, i = 1, \ldots, r$, be D-dimensional real vectors (greek indices $\alpha, \beta, \ldots$ run over the D internal coordinates), and introduce the $(n-r)$ linear combinations

$$y_k^\alpha = \sum_{i=1}^{r} \sum_{\beta=1}^{D} x_i^\beta J_{ik}^{\beta\alpha} \tag{56}$$

with $k = r+1, \ldots, n$ defined by the matrices J_{ik}.

The Gibbs weight of a configuration is

$$\mathrm{d}x \exp[-\mathcal{H}_J(x)] = \mathrm{d}^D x_1 \cdots \mathrm{d}^D x_r \mathrm{e}^{-\frac{1}{2\sigma^2}\sum_{i=1}^{r}|x_i|^2} \prod_{k=r+1}^{n} \delta(y_k, 0). \tag{57}$$

We consider the random model in which we have a probability measure over instances J_{ik}. Denote by $\mathrm{d}\mu_D$ the Haar measure over the group of special orthogonal transformations $SO(D)$. A possible choice is to take the variables J_{ik} as i.i.d. variables, distributed with $\mathrm{d}\mu_D$, i.e. decorrelated random rotations

$$\mathrm{d}\mu(J) = \prod_{i=1}^{r} \prod_{k=r+1}^{n} \mathrm{d}\mu_D(J_{ik}). \tag{58}$$

The problem has a large group of invariances. In the case $2r > n$, this allows us to reconduct the matrix J in a standard vectorial form. Three classes of manipulations on J, all unitary Jacobian, do not modify the value of the partition function:

- permutations of the columns:

$$J_{ik} \quad \longrightarrow \quad J_{i\sigma(k)} \qquad \sigma \in \mathfrak{S}_{n-r}$$

- orthogonal transformations on the rows:

$$J_{ik} \quad \longrightarrow \quad C_{ij} J_{jk} \qquad C \in SO(r)$$

- global rotations on internal space $\mathbb{R}^D$:

$$J_{ik}^{\beta\alpha} \quad \longrightarrow \quad R_{(i)}^{\beta\beta'} J_{ik}^{\beta'\alpha'} R_{(k)}^{\alpha\alpha'} \qquad R_{(j)} \in SO(D) \quad 1 \leqslant j \leqslant n.$$

As with probability 1 on our measure, $\mathrm{rank}(J) = \min(r, n-r) = n-r$, using these operations we can put the matrix J in the form

$$J' = \begin{pmatrix} Y \\ -I_{n-r} \end{pmatrix}.$$

This procedure of diagonalization corresponds to the algebraic resolution of the delta functions. In fact, the linear combinations (56) are

$$(y')_k^\alpha = \sum_{i=1}^{r} \sum_{\beta=1}^{D} x_i^\beta (J')_{ik}^{\beta\alpha} = \sum_{i=1}^{2r-n} \sum_{\beta=1}^{D} x_i^\beta Y_{ik}^{\beta\alpha} - x_{k-(n-r)}^\alpha. \tag{59}$$

Relabelling

$$w_k = x_{k+2n-r} \qquad k = 1, \ldots, n-r \tag{60}$$

we can write

$$w_k^\alpha(x, Y) = \sum_{i=1}^{2r-n} x_i^\beta Y_{ik}^{\beta\alpha} \qquad k = 1, \ldots, n-r. \tag{61}$$

Define the transposal matrix of homomorphisms Y^T as $(Y^T)_{ki}^{\alpha\beta} = Y_{ik}^{\beta\alpha}$. The Gibbs weight (57) can be rewritten as

$$\begin{aligned}\exp[-\mathcal{H}_J(x)] &= \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{i\leqslant 2r-n}|x_i|^2 + \sum_{k\leqslant n-r}|w_k(x,Y)|^2\right)\right)\\ &= \exp\left(-\frac{1}{2\sigma^2}\sum_{i\leqslant 2r-n} x_i^\alpha (I + YY^T)_{ij}^{\alpha\beta} x_j^\beta\right)\end{aligned} \tag{62}$$

which is of the form of the Gaussian model of section 9 (equation (50)), with $2r-n$ variables, $n-r$ constraints, and all the σ_i and the ρ_k equal to σ. The measure of integration $\mathrm{d}x$ stands for $\mathrm{d}^D x_1 \cdots \mathrm{d}^D x_{2r-n}$. With a slight abuse of notation, we can write the partition function of this model as

$$Z_{\text{Gauss}}(Y, \sigma) = \int \mathrm{d}x\, \mathrm{e}^{-\frac{1}{2\sigma^2} x^T (I_r + YY^T)x} \tag{63}$$

and, applying the duality transformation to this formulation of the problem, obtain

$$Z_{\text{Gauss}}(Y, \sigma) = \frac{(2\pi/\sigma)^{\frac{n}{2}}}{(2\pi)^{n-r}} Z_{\text{Gauss}}(-Y^T, \sigma^{-1}). \tag{64}$$

We could also apply the duality transformation directly to the original formulation. In this case we have r variables $x_1, \ldots, x_r$, with one-body functions $f_i(x_i) = \mathrm{e}^{-x_i^2/(2\sigma^2)}$ and $(n-r)$ interactions, encoded in the delta functions $g_k(y_k) = \delta(y_k, 0)$.

The dual problem has $(n-r)$ variables $\eta_1, \ldots, \eta_{n-r}$, with constant one-body functions $\widehat{g}_k(\eta_k) = \widehat{\delta}(\eta_k) = 1/(2\pi)$, and r interactions given by

$$\widehat{f}_i(\xi_i) = \sqrt{\frac{2\pi}{\sigma}} \exp\left(-\frac{\sigma^2 \xi_i^2}{2}\right) \qquad \xi_i^\beta = \sum_{k=r+1}^{n}\sum_{\alpha=1}^{D} \eta_k^\alpha (-J^T)_{ki}^{\alpha\beta}.$$

The definition of the transposal matrix is $(-J^T)_{ki}^{\alpha\beta} = -J_{ik}^{\beta\alpha}$. At this point, we can remove the trivial constant one-body terms. Then we can diagonalize the matrix $-J^T$ using the 'transposal' of the procedure described above for J (that is, moves on rows and columns are replaced by the corresponding moves on columns and rows). The partition function obtained in this way coincides with the right-hand side of (64).

In this case in which the entries of matrix J are non-trivial homomorphisms, we can check directly that, according with the general recipe of section 5, the entries $\widehat{J}_{ki}$ of the dual matroid are related to the *homomorphism-transposal* of the entries J_{ik} (in this case, the internal indices α and β are exchanged). Moreover, in agreement with the general description of section 7, we see how the resolution of delta constraints and the removal of constant factors are dual procedures.

11. A generalized Potts-like model

In this section we study the effects of the duality on a particular model which slightly generalizes the one we introduced in [13]. We deal with r variables $x_i \in \{0, \dots, q-1\}$ and a general Hamiltonian with $(n-r)$ Potts-like interaction terms provided by a matrix J

$$-\mathcal{H}(x; J, \{K_k\}, \{h_i\}) = \sum_{k=1}^{n-r} K_k \delta\left(\sum_i x_i J_{ik}\right) + \sum_{i=1}^{r} h_i \delta(x_i) \tag{65}$$

where the linear combinations in xJ and the delta functions are intended on $\mathbb{Z}_q$, that is $x \equiv x + nq$. The weights of the terms in the Hamiltonian are more conveniently expressed in terms of a vector $v \in \mathbb{R}^n$, defined as

$$\begin{aligned} v_i(x) &= \mathrm{e}^{h_i} - 1 \qquad \text{for} \quad i = 1, \dots, r \\ v_{r+k}(x) &= \mathrm{e}^{K_k} - 1 \qquad \text{for} \quad k = 1, \dots, n-r \end{aligned}$$

and of the functions

$$f_j(x) = 1 + v_j \delta(x). \tag{66}$$

If we introduce the matroid $B = (I_r|J)$, we can give the problem in the general form (19):

$$\exp\left(-\mathcal{H}_{\mathbb{Z}_q, B, \{f_j\}}(x)\right) = \prod_{j=1}^{n} [1 + v_j \delta((xB)_j)]. \tag{67}$$

We will also introduce an *ad hoc* notation for the partition function of this system:

$$Z_{\text{Potts}}(J; \{v_j\}) \equiv Z_{\mathbb{Z}_q, B, \{f_j\}} \tag{68}$$

where expressions (66) above for the $\{f_j\}$ are understood.

From the form of the Fourier transform of $f(x) = 1 + v\delta(x)$

$$\widehat{f}(\xi) = \frac{v}{q}\left(1 + \frac{q}{v}\delta(\xi)\right) \tag{69}$$

and with the convention of table 1 for the measure $\int_x$ in the case $X = \mathbb{Z}_q$, we have the following relation between the partition functions:

$$Z_{\text{Potts}}(-J^T; \{q/v_j\}) = \frac{\prod_{i=1}^{r} v_i}{\prod_{k=1}^{n-r} q/v_{r+k}} Z_{\text{Potts}}(J; \{v_j\}). \tag{70}$$

Let us specialize the general duality transformation we have obtained in some simple cases related to the model that we introduced in [13]. First of all, we will consider the case in which the magnetic field is constant ($v_j = v$ for all $j = 1, \dots, r$) and all the couplings are equal ($v_j = w$ for all $j = r+1, \dots, n$). Equation (70) in this case gives

$$Z_{\text{Potts}}(-J^T; q/w, q/v) = \frac{v^r}{(q/w)^{n-r}} Z_{\text{Potts}}(J; v, w). \tag{71}$$

We can give a simple mathematical interpretation to the limit $v \to 0$, $w \to \infty$ of the above formula. In the two dual models, the partition functions, properly regularized, just count the number of solutions of the linear equations $xJ = 0$ and $y(-J^T) = 0$. In this limit case the duality relation states that the rank of the matrix J is equal to the rank of $-J^T$, which is a trivial fact of elementary geometry. This model is exactly solvable for the ensemble of random matrices on a finite field, that is we know the exact probability distribution for the free energy and other physical quantities, and the duality relation indeed plays a role in the solution of the problem [13].

A particular case of (71) is when only $v \to \infty$ (and then $w' \to 0$) where again the partition functions should be properly regularized. The explicit expression is

$$\sum_x \delta(Jx)(1+w)^{-\sum_i(1-\delta(x_i))} = \sum_\eta \prod_k \left[\frac{w}{q} + \delta((-J^T y)_k)\right] \tag{72}$$

therefore, the duality relates a sum, restricted to the set of solutions of a linear system $Jx = 0$, where the difficulty of the problem lies in the external field contribution, to a sum, with no external field, on all the states of the spectrum, where the difficulty lies in the non-trivial correlations between the degeneracy of the energy levels. A quantitative study of this system will be done in a forthcoming paper [18].

We can use the general strategy of section 8 to investigate some sufficient conditions on the measure $\mu_0(J)$ for equation (71) to relate the partition functions of *the same* model at different values of the parameters v and w.

As the weight functions involved are of the form 'delta + constant', with no disorder parameters, the subgroup of automorphisms $A \subseteq \mathrm{Aut}(\mathbb{Z}_q)$ such that for a $a \in A$ we have $f(a(x)) = f(x)$ is isomorphic to the classical group $\mathbb{Z}_q^*$.[9] That is, if a is an integer coprime with q, the corresponding automorphism on $\mathbb{Z}_q$ acts as $a(x) = ax (\mathrm{mod}\, q)$.

The group of invariances obtained is sufficiently large that, following the strategy outlined in section 8, the self-duality condition can be verified for almost every reasonable ensemble of matrices. For example, this is the case for random matrices, for sparse matrices, both with nonzero elements chosen randomly among $\{1, \ldots, q-1\}$ or all equal to 1, and also for matrices with non-trivial correlations between the entries, if the correlations are symmetric under transposition, like the ensemble of incidence matrices of random bipartite graphs with vertices of fixed connectivity. An exception is the traditional ensemble for the study of XOR-SAT problems [19], which is a particular case of our model, but the matrices J are constrained to have a fixed number of nonzero entries per row, and do not have any constraint on the columns.

We note that, in the case of self-dual measure on the ensemble of matrices J and $v = w$, the coupling fixed point occurs for $v = \sqrt{q}$, which is indeed the critical point also for the ordinary Potts model on a geometrically self-dual planar graph. We will show in the next section how to recover self-duality explicitly in this case, in which other 'geometrical' manipulations are required.

12. Potts model on graphs and on planar graphs

The model discussed in the previous section is a generalization of the ordinary Potts model. In this section we will show how our duality relation can be reconstructed as the known duality of the Potts model on planar graphs [20, 7].

Consider a connected graph Λ, at this stage not necessarily planar, with sites $s \in \mathcal{S}$, bonds $b \in \mathcal{B}$ and elementary cycles (or *loops*) $l \in \mathcal{L}$. Assume that the number of sites, bonds and loops are respectively S, B and L. If we consider the graph as embedded on an orientable surface, the set $\mathcal{L}$ of the loops could be chosen to be the set of the plaquettes ('local' loops), plus the set of the generators of the first homology group, that is loops which turn around its handles. We remark that, at this point, this division is not necessary.

We will give an arbitrary orientation to the bonds and to the loops. That is, for each bond b, $s_{\mathrm{in}}(b)$ and $s_{\mathrm{out}}(b)$ will be respectively the first and second extrema of the bond. Furthermore,

[9] The group $\mathbb{Z}_q^*$ is defined as the integers in $\{1, \ldots, q-1\}$ which are coprime with q; the group operation is the product and the standard modulo q identification is understood.

each loop l will be associated with a B-dimensional vector $J_b(l)$, such that, having arbitrarily chosen an orientation for the loop, $J_b(l)$ is the number of times that the loop l runs over bond b with the same orientation, minus the times it runs over b in the opposite orientation.

The Potts model on the graph Λ is defined by the set of variables $\sigma_s \in \{0, \dots, q-1\}$ on the sites, and a set of interactions with coupling constants K_b on the bonds[10]. The Hamiltonian is

$$-\mathcal{H}_{\substack{\text{Potts}\\ \text{Graph}}}(\sigma; \Lambda, \{K_b\}) = \sum_{b\in\Lambda} K_b \delta\big(\sigma_{s_{\text{in}}(b)} - \sigma_{s_{\text{out}}(b)}\big). \tag{73}$$

If we introduce the functions

$$f_s(x) = 1 \tag{74a}$$

$$f_b(x) = 1 + v_b\delta(x) \qquad v_b = \mathrm{e}^{K_b} - 1 \tag{74b}$$

and the matroid $B = (I_S | J_{sb})$, with J an $S \times B$ matrix defined as

$$J_{sb} = \begin{cases} 1 & s = s_{\text{in}}(b) \\ -1 & s = s_{\text{out}}(b) \\ 0 & \text{otherwise} \end{cases} \tag{75}$$

the partition function

$$Z_{\mathbb{Z}_q, B, \{f_s, f_b\}} = \sum_{\sigma} \prod_{s} f_s(\sigma_s) \prod_{b} f_b\left(\sum_{s} \sigma_s J_{sb}\right) \tag{76}$$

is in the general form (20).

Now we remove the useless identity functions $f_s(x)$ and make the change of variables

$$x_b(\sigma) = \sigma_{s_{\text{in}}(b)} - \sigma_{s_{\text{out}}(b)}. \tag{77}$$

The weight of each configuration σ is now just $\prod_b f_b(x_b(\sigma))$. Next, we want to perform the sum of the partition function on the new variables x, instead of that on the old σ. To this end, we need to know how many configurations σ correspond to a certain configuration x. The answer is simple: they are exactly q if the circuitation of x on each loop is zero (that is $\sum_b J_b(l) x_b \equiv 0 \bmod q$),[11] and zero if the circuitation is nonzero for some loop. So we have

$$\sum_{\sigma} \quad \longrightarrow \quad q \sum_{x} \prod_{l\in\mathcal{L}} \delta\left(\sum_{b} J_b(l) x_b\right) \tag{78}$$

and, with the definitions

$$f_b(x) = 1 + v_b\delta(x) \qquad f_l(x) = \delta(x) \tag{79}$$

$$J_{bl} = J_b(l) \qquad B' = (I_B | J_{bl}) \tag{80}$$

we have again a system in the general form (20). At this point we can perform the duality transformation. According to the general formula (21), and to the expressions for the Fourier transforms of $f_b(x)$ and $f_l(x)$, respectively

$$\widehat{f}_b(\xi) = \frac{v_b}{q}\left(1 + \frac{q}{v_b}\delta(\xi)\right) \qquad \widehat{f}_l(\xi) = 1 \qquad \widehat{B'} = (-J_{lb} | I_L) \tag{81}$$

[10] We introduce an additive group structure $\mathbb{Z}_q$ for the q 'colours' of the model, although ordinary Potts model Hamiltonian has a $\mathfrak{S}_q$ total permutational symmetry between the colours. We stress again the important fact that in general the symmetry group of the model *is not related* to the group introduced to perform the Fourier transform.

[11] More generally, for Λ not connected, we have a factor q for each connected component.

the result of the transformation is a system in the original general form (76). If the matrix $-J_{lb}$ is of the same kind as J_{sb}, with exactly two nonzero entries per column, one equal to $+1$ and the other equal to -1, we can reconstruct a dual graphical version of the original Potts model. As shown in detail in appendix A, this always happens when the original graph is planar. In brief, when a graph is planar, every bond appears in exactly two loops (the plaquettes on the two sides). If we choose a uniform orientation for the loops (all clockwise or all anticlockwise), it appears in the two loops with opposite sign. So, in this dual model, two neighbouring plaquettes of the original graph appear to interact with the dual coupling constant of the common bond ($v'_b = q/v_b$). If we want to redraw a graph such that the system has the variables on the sites and the interactions on the bonds, the desired graph is exactly the dual graph of the original one, in the sense of cell-complexes duality.

The procedure of this section can be applied with minor modifications to obtain the XY-model–Coulomb gas relation [21, 22], provided that the XY-model Hamiltonian is in the Villain form. Essentially, the only difference is that the couple of dual groups is $(\mathbb{Z}_q, \mathbb{Z}_q)$ for the Potts model and $(U(1), \mathbb{Z})$ for the XY-model–Coulomb gas relation.

13. The coupling with gauge fields

In this section we describe the application of the procedure described in section 6 to a well-known physical situation. The abstract framework that we introduced should now assume a more transparent meaning: that procedure mimics what happens if we introduce a new set of gauge fields, coupled with the terms of interaction, and then a new term of the Hamiltonian depending on the physical content of these new fields.

The paradigm of this procedure is the introduction of minimal electrodynamic interaction in a scalar continuous field theory, with the Lagrangian

$$\mathcal{L}_0 = \int \mathrm{d}\phi [V(\phi) + \partial_\mu \phi \partial^\mu \phi]. \tag{82}$$

The derivative operator ∂_μ has to be modified into a covariant derivative operator D_μ, defined by the action on the fields as $D_\mu \phi = \partial_\mu \phi - A_\mu$. Then we have to add the dynamics of the gauge field, $F_{\mu\nu}F^{\mu\nu}$, where the field tensor is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The new Lagrangian will be

$$\mathcal{L}_{\mathrm{QED}} = \int \mathrm{d}\phi \, \mathrm{d}A_\mu \left[V(\phi) + D_\mu \phi D^\mu \phi + \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} \right]. \tag{83}$$

If we consider the statistical-mechanics model whose Hamiltonian is derived from the lattice discretized version of this model (as is done, for example, in [23]), we replace partial derivatives $\partial_\mu \phi(x)$ with finite difference operators $(\partial\phi)_{n;\mu} = \phi_{n+\widehat{\mu}} - \phi_n$, so that, as usual, the interactions lie on the bonds of the lattice. The linear combinations $(\partial\phi)_{n;\mu}$ are the arguments of the interaction terms in the Gibbs factor.

The discretized gauge fields $A_{n;\mu}$ couple to the terms of interactions in such a way that the finite differences of the fields, $(\partial\phi)_{n;\mu}$, are modified into the covariant derivative $(D\phi)_{n;\mu} = (\partial\phi)_{n;\mu} - A_{n;\mu}$.

The discretized field tensor $F_{n;\mu\nu} = A_{n;\mu} + A_{n+\widehat{\mu};\nu} - A_{n+\widehat{\nu};\mu} - A_{n;\nu}$ takes the circuitation of the vector field $A_{n;\mu}$ around each elementary plaquette $(n;\mu\nu)$ of the lattice, and thus is a linear combination of the entries $A_{n;\mu}$.

The current quadrivector is given by the divergence of the electromagnetic tensor, $j_{n,\mu} = \sum_\nu (F_{n,\mu\nu} - F_{n-\nu,\mu\nu})$, and its divergence, which is automatically zero, is given by $(\nabla \cdot j)_n = \sum_\mu (j_{n,\mu} - j_{n-\mu,\mu})$.

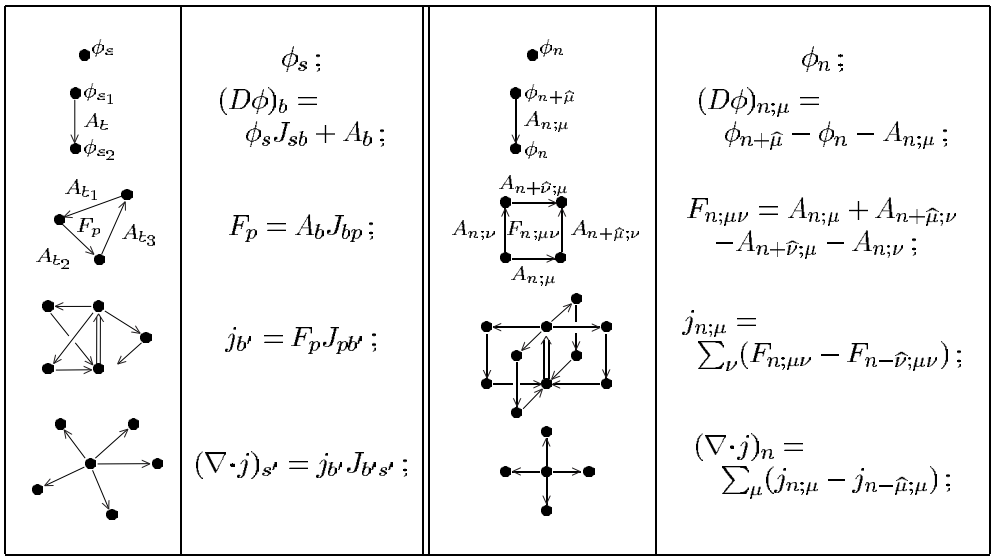


Figure 1. A table which collects the most important quantities in discretized versions of QED, for regular lattices (right), and for arbitrary graphs (left).

We can generalize this frame to an abstraction of the geometric structure of a lattice, in which we have a set of sites $\{s\}$, a set of bonds $\{b\}$, a set of plaquettes $\{p\}$ and a set of dual sites $\{s'\}$. In the previous case we would have for the sites $\{s\} = \{n\}$, for the bonds $\{b\} = \{(n, \mu)\}$, for the plaquettes $\{p\} = \{(n, \mu\nu)\}$ and for the dual sites $\{s'\} = \{(n, \mu\nu\rho)\}$.

The action of gradient and divergence operators is now performed via some adjacent matrices. In particular we will introduce three matrices J_{sb}, J_{bp} and $J_{ps'}$, such that, given the two sets respectively of scalar fields ϕ_s and of gauge fields A_b, we have

$$(D\phi)_b = \phi_s J_{sb} - A_b \qquad F_p = A_b J_{bp} \qquad (\nabla \cdot j)_{s'} = F_p J_{ps'}. \tag{84}$$

The geometric description of these quantities, and a comparison with regular-lattice analogues is given in figure 1.

We will be interested in the theory with the Gibbs weight

$$\exp(-\mathcal{H}_{\mathrm{QED}}(\phi, A)) = \prod_s f_s(\phi_s) \prod_b f_b((D\phi)_b) \prod_p \widehat{g}_p(F_p) \prod_{s'} \widehat{g}_{s'}((\nabla \cdot j)_{s'}) \tag{85}$$

with the four sets of functions given by

$$f_s(x) = 1 \qquad f_b(x) = \exp\left(-\frac{1}{2}x^2\right) \tag{86a}$$

$$\widehat{g}_p(x) = \exp\left(-\frac{1}{4e^2}x^2\right) \qquad \widehat{g}_{s'}(x) = \delta(x). \tag{86b}$$

In our general frame of section 4 this model corresponds to the matroid

$$B = \begin{pmatrix} I & -J_{sb} & & \\ & I & J_{bp} & J_{bp}J_{ps'} \end{pmatrix}$$

which is in the general form described in section 6. Following the procedure described therein, we obtain for the dual matroid

$$\widehat{B} = \begin{pmatrix} I & -J_{s'p} & & \\ & I & -J_{pb} & -J_{pb}J_{bs} \end{pmatrix}$$

in which sites are exchanged with dual sites, bonds are exchanged with plaquettes and the relative orientation of bonds and plaquettes is reversed (in the geometrical duality for three-dimensional lattices, this corresponds to the fact that the orientations of bonds and dual bonds change in such a way that the vector product changes parity).

Introducing the *ad hoc* notation for the partition function

$$Z_{\mathrm{QED}}(e, B) = \int \mathrm{d}\phi_s \, \mathrm{d}A_b \exp\left(-\frac{1}{2}\sum_b (D\phi)_b^2 - \frac{1}{4e^2}\sum_p F_p^2\right) \prod_{s'} \delta((\nabla \cdot j)_{s'}) \tag{87}$$

with the proper normalizations of the integrals, the duality relation states that

$$Z_{\mathrm{QED}}(e, B) = \frac{(2\pi)^{\frac{r}{2}}}{(4\pi e^2)^{\frac{n-r}{2}}} Z_{\mathrm{QED}}\left(\frac{1}{2e}, \widehat{B}\right) \tag{88}$$

where, as always, r is the rank of B, and in our case is equal to the number of sites plus the number of bonds, and $(n - r)$ is the rank of $\widehat{B}$, and in our case is equal to the number of dual sites plus the number of plaquettes. Again we see how duality exchanges a regime of high coupling with a regime of low coupling.

A concrete example could be the one of a cubic three-dimensional lattice, with the ordinary notions of sites, bonds and plaquettes, and the dual sites being the elementary cubic cells of the lattice. The geometric duality applied to the lattice naturally exchanges sites with dual sites and bonds with plaquettes. In this case, the presence of the delta function for the terms of current divergence is redundant, as the constraint is automatically satisfied.

Random-system variants could involve a measure $\mu(B)$ over an ensemble of random tessellations of the three-dimensional space, or, more generally, over an ensemble in which adjacency matrices are not directly derived from a geometrical structure, and the condition $\nabla \cdot j = 0$ really needs to be enforced via a delta function in the Gibbs weight.

In the case in which, following the recipe of section 8, we have a self-dual measure $\mu^*(B)$, the fixed point of the involution for the electric charge e is at the value $2e^2 = 1$.

14. Conclusions

The notion of duality as given by Kramers and Wannier, after its original introduction in 1941, has been applied to a wide range of periodic finite-dimensional systems of statistical mechanics. In contrast, little has been done in the direction of a systematic description of duality for a class of models sufficiently large to include, for example, mean-field disordered systems.

Our idea in this paper is that, in the Kramers and Wannier formulation, a sort of two-step procedure is implicit: an *algebraic duality*, which exchanges variables and interactions, and performs a Fourier transformation on the terms of the Gibbs weight, and a *geometric duality*, which, exploiting the geometrical structure of the lattice, allows for a reinterpretation of the dual system in a formulation which, in some contexts, is suitable for a direct comparison with the original system.

In the wider context of disordered systems the second step is in general impossible. Nonetheless, in many cases the first step is already suitable for a fruitful comparison of the two systems related by duality.

The general recipe is the following: given some *variables* and some *interactions*, valued on some Abelian group G, the Gibbs weight depends on the two ingredients:

- two sets of functions, weighting respectively the original variables and the auxiliary variables corresponding to some linear combinations of the original ones (the *interactions*);

- the specific pattern of linear combinations of the variables, encoded in a matrix J of group homomorphisms.

The dual system exchanges variable functions and interaction functions with the Fourier anti-transform and transform of interaction functions and variable functions, and the new pattern is given by the matrix $-J^T$ of transposed homomorphisms.

We give some concrete applications for specific simple models: a random Gaussian model, a random Potts-like model, a random variant of discrete scalar QED, ... For each of these systems, a short discussion over the duality relation is proposed, and a hint over self-duality conditions is suggested. Actually, in the cases of interest, this discussion is a simple specialization of a general discussion done in section 8, and analogous to the one already presented in [13], where the duality, in its first application, was called *variable-clause duality*.

An unexpected connection with matroid theory has emerged in the analysis of the subject. The *a posteriori* justification for this fact is that the pattern of interaction which is involved in the definition of the Gibbs weight is intrinsic with respect to a choice of basis for the space of configurations. Matroid theory, in simple words, describes mathematical objects which convey basis-intrinsic information on matrix-like objects, like the pattern of linear dependence between the columns of a matrix. Hence, it is small surprise if the matroid related to the dual system is the matroid-dual of the matroid related to the original system.

In a forthcoming paper we will discuss a further generalization of this framework, in which the crucial concepts of *Fourier transform* and of *group homomorphism* for Abelian groups are generalized to different integral transform and natural notions of homomorphism, applied to mathematical structures different from Abelian groups. A first case is the one in which, for partially ordered sets, the natural notion of transform is the Möbius transform, and the natural notion of homomorphisms is related to the theory of Galois connections [24].

Acknowledgments

We thank Alan Sokal for introducing us to the relevance of the theory of matroids for Potts-like systems and for valuable discussions in the reading of the manuscript. We are grateful to F Y Wu who attracted our attention to previous relevant results that we had ignored.

Appendix A. Duality for graphical matroids and planar graphs

An important subset of regular matroids[12] is the set of graphical matroids. Many results exist on the subject (cf, for example, [17], chapter 5), derived in the spirit of generality characteristic of matroid theory. In this appendix we will show some constructive procedures which apply to particular classes of graphical matroids, so in this sense they are less general, but also less demanding restatements of known facts. Hopefully, they will be useful to understand how to interpret the classical formulations of dualities for models on planar lattices inside our general setting for duality.

A precise terminology will be introduced below. Here we summarize the main results:

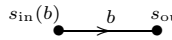
Theorem 1. *The 'site-bond' matroid J_{sb} and the 'loop-bond' matroid J_{lb} of a graph $\mathcal{G}$ are dual matroids.*

Theorem 2. *The 'site-bond' matrix J'_{sb} of a planar graph $\mathcal{G}$ is equal to the 'loop-bond' matrix J'_{lb} of the corresponding geometrical dual graph $\mathcal{G}^*$, and conversely the 'loop-bond' matrix of $\mathcal{G}$ is equal to the 'site-bond' matrix of $\mathcal{G}^*$.*

[12] That is, matroids which are representable over every field.

A.1. Matroid duality for graphical matroids

Consider a graph $\mathcal{G}$, with vertices (or sites) $s \in V$ and edges (or bonds) $b \in E$. We give an arbitrary orientation to the bonds. That is, for each bond b, $s_{\text{in}}(b)$ and $s_{\text{out}}(b)$ will be respectively the first and second extrema of the bond



. (A.1)

Define a 0-form φ as a function from V to a field $\mathbb{F}$ and a 1-form ψ as a function from E to $\mathbb{F}$. The operators ∂ and ∂^* transform 0-forms into 1-forms and vice versa with the following definition:

$$\partial\varphi(b) = \varphi(s_{\text{in}}(b)) - \varphi(s_{\text{out}}(b)) \tag{A.2}$$

$$\partial^*\psi(s) = \sum_b \psi(b)\big(\delta_{s,s_{\text{in}}(b)} - \delta_{s,s_{\text{out}}(b)}\big). \tag{A.3}$$

Intuitively, they correspond to the discretization of the gradient operator on scalar fields and to the divergence operator on vector fields, generalized to arbitrary lattices.

The sites are implicitly a particular case of 0-form, $\varphi_{(s)}(s') = \delta_{s,s'}$. The set of 0-forms is a vector space over $\mathbb{F}$, of dimension $|V|$. It is naturally decomposed in two vector spaces, $\mathcal{S}$ and $\mathcal{C}$, where 0-forms in $\mathcal{C}$ are constant on each of the C connected components of $\mathcal{G}$ (so $\dim(\mathcal{C}) = C$) and 0-forms in $\mathcal{S}$ are obtained by quotient (so $\dim(\mathcal{S}) = S = |V| - C$). Projection on $\mathcal{S}$ is obtained via the operator $\frac{1}{2}\partial^*\partial$.

Analogously, the set of 1-forms is a vector space $\mathcal{B}$ over $\mathbb{F}$, of dimension $B = |E|$. The bonds are a particular case of generic 1-forms, where it is understood $\psi_{(b)}(b') = \delta_{b,b'}$, and are a basis for the space $\mathcal{B}$. We will define the loops as the 1-forms ψ such that $\partial^*\psi = 0$. Elementary plaquettes are particular cases of loops: if ∂p is the border of an oriented plaquette p, $\psi_{(p)}(b)$ is the characteristic function on ∂p. The set of loops is a vector space $\mathcal{L}$ over $\mathbb{F}$, of dimension L.

Note that $S + L = B$ as a consequence of the Euler formula. More deeply, the vector space $\mathcal{B}$ is decomposed into $\partial\mathcal{S} + \mathcal{L}$, that is, a generic 1-form ψ is uniquely decomposed into $\psi = \partial\varphi + \chi$, with $\varphi \in \mathcal{S}$ and $\chi \in \mathcal{L}$ (Hodge decomposition).

The patterns of linear dependence of the set of bonds projected onto the vector spaces $\partial\mathcal{S}$ and $\mathcal{L}$ are matroids. They will be called respectively the *site-bond* and the *loop-bond* matroids. Given a choice of basis for $\mathcal{S}$ and $\mathcal{L}$, we will denote the corresponding vectorial representations as J_{sb} and J_{lb}. That is, for a set of 0-forms $\{\varphi_s\}$ chosen as a basis for $\mathcal{S}$, the entries of J_{sb} will be

$$J_{sb} = \partial\varphi_s(b) \tag{A.4}$$

and for a set of 1-forms $\{\psi_l\}$ chosen as a basis for $\mathcal{L}$, the entries of J_{lb} will be

$$J_{lb} = \psi_l(b). \tag{A.5}$$

We will give a particular skill choice of basis, for which the two matroids are both in standard vectorial form, and manifestly dual.

Assume $\mathcal{G}$ is connected (the argument easily generalizes): choose a 'ground' site $s_0 \in V$ and a spanning tree $\mathcal{T} \subseteq \mathcal{G}$, and orient the bonds such that the open path on $\mathcal{T}$ connecting s_0 to s is oriented from s_0 to s. For each site $s \neq s_0$ there is exactly one in-going bond in $E(\mathcal{T})$. Label this bond with b_s. Say that $s \succ s'$ iff an oriented path $\gamma_{s'\to s}$ in $\mathcal{T}$ exists. Choose a basis for $\mathcal{S}$ as

$$\varphi_s(s') = \begin{cases} +1 & s \succ s' \\ 0 & \text{otherwise.} \end{cases} \tag{A.6}$$

So we have

$$\partial\varphi_s(b_{s'}) = \delta_{s,s'} \qquad \partial\varphi_s(b_l) = \begin{cases} +1 & s \succ s_{\text{in}}(b_l) \quad s \not\succ s_{\text{out}}(b_l) \\ -1 & s \succ s_{\text{out}}(b_l) \quad s \not\succ s_{\text{in}}(b_l) \\ 0 & \text{otherwise.} \end{cases} \tag{A.7}$$

We will choose a basis for $\mathcal{L}$ such that for each bond $b \in E \setminus E(\mathcal{T})$ there is exactly one 1-form ψ_l such that $\psi_l(b)$ is nonzero. So we can label the edges in $E \setminus E(\mathcal{T})$ as b_l. More precisely, the 1-form ψ_l corresponds to the only self-avoiding closed path on $\mathcal{T} \cup \{b_l\}$, oriented according to b_l. Analytically this can be expressed as

$$\psi_l(b_{l'}) = \delta_{l,l'} \qquad \psi_l(b_s) = \begin{cases} -1 & s \succ s_{\text{in}}(b_l) \quad s \not\succ s_{\text{out}}(b_l) \\ +1 & s \succ s_{\text{out}}(b_l) \quad s \not\succ s_{\text{in}}(b_l) \\ 0 & \text{otherwise.} \end{cases} \tag{A.8}$$

It can immediately be checked that J_{sb} and J_{lb} are in standard vectorial form, respectively with the identity block on the left-hand side and on the right-hand side. From the observation that $\partial\varphi_s(b_l) = -\psi_l(b_s)$ we obtain that they are dual in the matroid sense:

$$J_{sb} = (I, Y_{sl}) \qquad J_{lb} = \left(-Y_{ls}^T, I\right). \tag{A.9}$$

As we have found an explicit choice of basis for which the standard forms of the site-bond matroid and of the loop-bond matroid are dual, we have implicitly proved that the matroids are dual, which is the claim of theorem 1.

A.2. Graphical duality for planar graphs

In the previous section we have seen how the 'abstract' site-bond and the loop-bond matroids for a given graph are dual in the sense of matroid duality. This is true in general, as the matroid definition is intrinsic, and specifically we have verified it in a particular choice of basis, for which both the matroids are in standard form.

In this section we show the result of theorem 2: that the site-bond and the loop-bond incidence matrices for a connected planar graph and for its graphical dual are exchanged by the graphical duality. This result relies on a particular choice of the loops as the elementary plaquettes of the lattice, which is more natural for the case of planar graphs, but in this sense is more accidental.

A graph $\mathcal{G}$ is said to be *planar* when it is embeddable on a genus-0 surface with no crossings between the bonds. Consider the case in which $\mathcal{G}$ is connected. The classical Euler relation for this kind of graph claims that $V + F = B + 2$, where V is the number of sites, B is the number of bonds and F is the number of faces.

One can immediately define the site-bond and loop-bond incidence matrices, A_{sb} and A_{lb}, for non-oriented bonds and loops. Just state that A_{sb} is 1 if the site s is an extremum of the bond b and 0 elsewhere, and A_{lb} is 1 if the bond b is a side of the face l and 0 elsewhere.

The graphical dual $\mathcal{G}^*$ is defined by the following procedure. Replace each plaquette l by a dual vertex s^* (white bullets in figure A1), each bond b by a dual (dashed) bond b^*, which crosses it and connects the dual vertices corresponding to two neighbouring plaquettes, and each site s with the dual plaquette l^* delimited by the dual bonds of the bonds incident on the site. It is clear that

$$A_{s^*b^*}(\mathcal{G}^*) = A_{lb}(\mathcal{G}) \qquad A_{l^*b^*}(\mathcal{G}^*) = A_{sb}(\mathcal{G}). \tag{A.10}$$

If we put an orientation on the bonds and a direction of rotation on the faces, the matrices A_{sb} and A_{lb} will be promoted to matrices J'_{sb} and J'_{lb}, in which some minus signs appear. Because

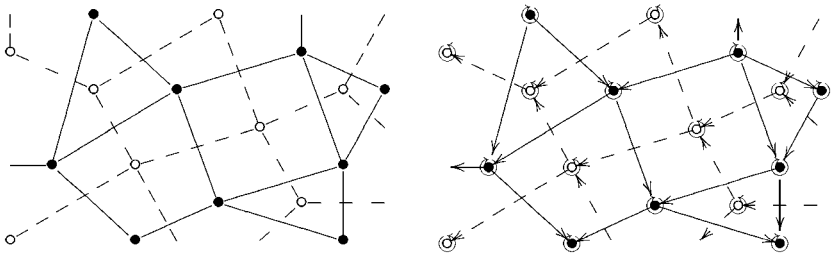


Figure A1. Graphical dual for a planar graph (left), and a consistent choice of orientations for the bonds and directions of rotations for the faces for the relation (A.11) to hold (right).

of the planarity of the graph, we could give a recipe such that relation (A.10) is promoted to a relation

$$J'_{s^*b^*}(\mathcal{G}^*) = J'_{lb}(\mathcal{G}) \qquad J'_{l^*b^*}(\mathcal{G}^*) = J'_{sb}(\mathcal{G}). \tag{A.11}$$

The recipe is the following: choose the direction of rotation to be the same for all the faces, (for example clockwise), and opposite between faces and dual faces. Choose the orientation of the bonds arbitrarily. The orientation of the dual bonds must be such that the vector product between each bond and its dual bond has the same sign with respect to the normal versor outgoing from the surface.

Both the matrices J'_{sb} and J'_{lb} are not of maximal rank, and the only linear relation is given by the sum of all the rows with equal coefficients. This accounts for the offset of 2 in the Euler formula $V + F = B + 2$. In fact the number of 'independent sites' in the loop-bond matroid (0-forms up to an arbitrary constant) is $S = V - 1$ and the number of independent loops is $L = F - 1$, from which we recover the relation $S + L = B$.

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